— Exercises —

1. **Useful fact.** For which α does the following integral converge:
   (a) $\int_0^1 t^\alpha dt$
   (b) $\int_1^\infty t^\alpha dt$
2. **A difference between improper integrals and integrals on a compact set** Let f be the function defined on $[0,\infty[\times[0,\infty[$ by $f(x,t) = xe^{-xt}$. Show by direct computation that, for every $x \geq 0$, $G(x) = \int_0^\infty f(x,t)dt$ converges. Is the function G continuous on $\mathbb{R}_{\geq 0}$?
3. **A not absolutely convergent integral.** Show that the integral $\int_\pi^\infty \frac{sin(t)}{t} dt$ converges (*hint: use 13.2.7 or an integration by parts*) but $\int_\pi^\infty \left|\frac{sin(t)}{t}\right| dt$ does not converge (*hint: for* $t \in [k\pi, (k+1)\pi]$, $\frac{1}{t} \geq \frac{1}{(k+1)\pi}$).
4. **The Gauss integral.** Show that $\int_{-\infty}^\infty e^{-t^2} dt$ converges. *Hint: use that* $t^2e^{-t^2} \xrightarrow{t\to+\infty} 0$.

— Problems —

5. **A difference between improper integrals and series of numbers.** Let f be a function defined, continuous and integrable on $\mathbb{R}_{\geq 0}$. We assume that f is decreasing on $\mathbb{R}_{\geq 0}$. Show that $f(t)$ tends to 0 when t tends to 0. Is it still true when f is not assumed to be decreasing?
6. **The Gamma function.** For every $x > 0$, we put $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$.
   (a) Show that $\Gamma(x)$ converges for every $x > 0$. Is the integral convergent for $x = 0$?
   (b) Prove that the resulting function Γ is continuous on $\mathbb{R}_{>0}$.
   (c) Prove that for every $x > 0$, we have $\Gamma(x+1) = x\Gamma(x)$. Using this fact, explain why $\Gamma(n+1) = n!$.
   (d) Show that $\Gamma(\frac{1}{2}) = \int_{-\infty}^\infty e^{-t^2}dt$. Assuming that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, find a formula for $\Gamma(\frac{2n+1}{2})$ for every n.
7. **Another difference between improper integrals and integrals on a compact set.**
   Let $(f_n)_n$ be a sequence of integrable functions $[0,+\infty) \to \mathbb{R}$ converging uniformly to f. Show that f need not be integrable on $[0,+\infty)$. Show that even when f is integrable, it may happen that the limit of integrals is not the integral of the limit.